Journal of Statistical Physics, Vol. 122, No. 6, March 2006 (© 2006)
DOI: 10.1007/s10955-006-9073-3

Pulled Fronts and the Reproductive Individual

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Received December 24, 2005; accepted January 25, 2006
Published Online: March 23, 2006

In *War and Peace*, Leo Tolstoy expressed his opinions about the importance of "great men" in determining the outcome of historical processes. He said that particular individuals are relatively unimportant in comparison to the overall "forces of history", comprising the actions, ideas, and emotions of the masses of human beings. Amusingly enough, the relative importance of the individual and the mass has become an interesting and important distinction in the development of theories related to front propagation in condensed matter systems. Several pieces of recent work focus upon situations in which the individual may count for a lot, and then give surprising results about the consequences of this individual importance

Focus upon a system with two phases separated by a front. If one phase is more stable than the other, we can see a motion in which the stable phase propagates into the region previously occupied by the less stable one. One example arises in the theory of population dynamics. In one case, a more robust population moves into territory that was occupied by a less "fit" population. This description can fit a problem which might equally be catch the attention of a biologist, historian, physicist, or mathematician.

There are two possibilities for the propagation of a one-dimensional front: Either the motion is "pulled" by the growing bits of stable phase extending far into the unstable region or alternatively it is "pushed" by the growth which occurs within the bulk of the boundary region.(13) If the propagating phase were a population of humans we would say the growth was pulled by the individuals at the very edge of the frontier or, conversely, that it was pushed by population pressure in the bulk.

As pointed out in detail in an extensive recent review by Wim van Saarloos,(13) the distinction between the two cases corresponds to a difference in behavior. The pulled case is much more delicate, subject to fluctuations, and responsive to

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small effects at the frontier. Two recent papers[4,5] deal with the pulled case, and emphasize this delicacy. Here I shall describe these papers and other recent work, emphasizing the qualitative aspects of the situation. But, first I should describe the basics.

The classic pulled situation is described by the equation of R. A. Fisher[6] and A. Kolmogorov and coworkers.[8] The equation,

$$\frac{\partial \rho}{\partial t} = \nabla^2 \rho + \rho(1 - \rho), \tag{1}$$

is denoted in Ref. (13) as F-KPP. Here one can view ρ as a dimensionless population density, with the value unity representing the saturation density. The change of the density in time, t, occurs because of a diffusion through space–the first term on the right– and also because of "natural" birth and death processes– the second term on the right. In Equation (1), F-KPP takes the net population growth to vanish in the absence of a population ($\rho = 0$) and to approach zero when the population density reaches its saturation value ($\rho = 1$). These two population densities represent the two stationary, time independent, states of the system, with the former solution being unstable, the latter stable.

A classic analysis[1] describes the behavior of a frontier separating two the two different phases. Take an ansatz of the form

$$\rho(\mathbf{r}, t) = f(x - vt) \tag{2}$$

with f going to zero as x goes to plus infinity and going to one as x goes to minus infinity. There are a family of solutions of this sort, depending on a parameter, λ, that determines how fast f approaches zero as x goes to infinity. The solutions have, for large x,

$$f(x) \to \text{const}\ e^{-\lambda x}. \tag{3}$$

There is a solution like this for each sufficiently small value of λ. The velocity is determined by the linearized behavior for large x and small ρ

$$v(\lambda) = \lambda + 1/\lambda \tag{4}$$

Note that this has a minimum, $v = v^* = 2$, for $\lambda = \lambda^* = 1$. For values of $\lambda < \lambda^*$, there are a continuum of solutions depending upon the large-x fall-off of the initial data. Basically the smallest value of λ in the asymptotic falloff determines everything. Alternatively, for initial data which falls off very rapidly for large x, all solutions approaches the one with velocity equal to 2 and $\lambda = 1$.

This interesting behavior of the spreading stable region has attracted very considerable attention. Recently, attention has focused upon the peculiar sensitivities and slow convergence of these pulled fronts. For example, the actual front given by the F-KPP equation lags behind a front moving with uniform velocity v^* by an amount which is logarithmic in the time after formation of the front.[3]

Additional sensitivity can be seen if one makes the growth rate in Equation (1) vary with position. For example, in a series of papers Levine and coworkers make that term grow linearly with x, as if the landscape were getting more fertile as one moves in that direction.[4,9,11] In a model defined by the equation

$$\frac{\partial \rho}{\partial t} = \nabla^2 \rho + (1 + \epsilon x)\rho(1 - \rho), \tag{5}$$

with positive ϵ, one finds a front velocity which continually accelerates as it moves toward larger x. Worse yet, the velocity never really settles down to an asymptotic behavior. Instead it retains a very considerable sensitivity to initial data.

Some of this sensitivity disappears if one goes from a model involving a continuous population density ρ, to a "grainy" model involving many individuals with a saturation at N individuals per unit length. In this discrete case, the solution of Equation (5) gives a front velocity that depends upon N. In one region of behavior, this velocity is proportional[4] to $(\ln N)^{1/3}$.

This logarithmic–hence very slow– convergence for large N is characteristic of pulled models. Even in the grainy version of the F-KPP model of Equation (1), the velocity contains a correction[2] proportional to $1/(\ln N)^2$. In contrast, in pushed models[10] the characteristic behavior of the velocity is to have a much smaller graininess correction, one proportional to some inverse power of N.

The importance of the individual on the frontier is shown even more strikingly in the paper of Brunet and Derrida.[5] Here there is once more a a pulled front and a discretization involving N particles per unit of length. Now the particles exist in a noisy environment with a noise of order $N^{-1/2}$. The main result of the paper is that the front velocity is determined by the noise on the last particle, standing at the far frontier.

Indeed, both historians and ecologists[12] have noticed that it is often the populations at the frontier that count. Of course there are many situations and models in which the fronts are pushed and well placed individuals count for less. Dynamical or historical[7] systems can do lots of different things. It all depends.

ACKNOWLEDGMENTS

This is a revision of a note originally posted on the web page of the Journal Club for Condensed Matter Physics http://www.bell-labs.com/jc-cond-mat/. I have had useful conversations on the topics considered here with Herbert Levine, Bernard Derrida, and Wim van Saarloos. This research is supported in part by the Materials Laboratory at the University of Chicago and was also supported by a very productive visit to the Lorentz Institute at Leiden. I thank that institute and Dr. van Saarloos for their hospitality.

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